Disjoint congruence classes and a timetabling application

Geir Dahl

Center of Mathematics for Applications, University of Oslo, P.O. Box 1053 Blindern, NO-0316 Oslo, Norway, and Institute of Transport Economics, N-0349 Oslo, Norway. Email: geird@math.uio.no

Abstract

We consider a combinatorial problem motivated by a special simplified timetabling problem for subway networks. Mathematically the problem is to find (pairwise) disjoint congruence classes modulo certain given integers; each such class corresponds to the arrival times of a subway line of a given frequency. For a large class of instances we characterize when such disjoint congruence classes exist and how they may be determined. We also study a generalization involving a minimum distance requirement between congruence classes, and a comparison of different frequency families in terms of their "efficiency". Finally, a general method based on integer programming is also discussed.

Keywords: disjoint congruences, timetabling, packing.

1 Introduction and the main problem

We consider a combinatorial problem involving disjoint congruence classes. The problem is motivated by an application in subway timetabling. Although our model is simplified, it is still of some practical interest. Literature on more realistic scheduling problems is discussed below.

Motivation. Consider a subway network where vertices and edges correspond to stations and direct links between stations, respectively. Each subway line corresponds to a path between two vertices in this network; an origin and a destination. We shall consider a timetabling problem in a simple network. In [6] timetabling problems for complex networks are treated and requirements concerning the resulting vehicle schedules are taken into account. In the present paper we consider a simplified problem involving the strategic decision on which frequencies to use for the different lines. We assume that there is a bottleneck in the network, e.g. a tunnel in the city center, through which all lines must pass. For instance, in Oslo, a single tunnel connects the eastern and the western side of the city (in this tunnel there are several stations all along a path). We only consider the westbound lines as eastbound lines may be treated similarly (and independently using a suitable stop at the terminal stations). We also assume fixed travel times between stations. With these assumptions, a schedule for a line is completely specified by its arrival times at, say, the Central Station which is one of the stops in the tunnel. Due to a gradual increase of traffic it is of interest to traffic authorities to consider adding new lines or changing the frequency of present lines. This motivates the question: *which combinations of lines and frequencies are feasible?* This is the problem discussed in this paper, considered from a mathematical viewpoint.

To be fair, we should say that the schedule in Oslo has been very simple, with all lines operating in 15 minute periods. Recently the frequency was doubled on one of these lines in the morning rush (called a "7/8 minute line"). In any case, we think the mentioned problem has interesting mathematical properties and is worth a study.

The mathematical model. We consider periodic schedules, and the *period* p of a line, the time between consecutive trains of that line, is a divisor of 60, so that the schedule repeats every hour. The *frequency* of a line with period p is $f = 60/p$; the number of arrivals during an hour. For instance, if the period of a line is 15 then its arrival times at the Central Station could be

$$\ldots \bullet\ 14.01 - 14.16 - 14.31 - 14.46\ \bullet\ 15.01 - 15.16 - 15.31 - 15.46\ \bullet \ldots$$

Let $\mathcal{P}$ be the possible periods so $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$. We consider n lines, indexed by $1, 2, \ldots, n$ and the corresponding *period-vector* is $p = (p_1, p_2, \ldots, p_n) \in \mathcal{P}^n$. We may assume that p is nondecreasing, i.e., $p_1 \leq p_2 \leq \cdots \leq p_n$. It may be convenient to write p differently to indicate equal components

$$p = (p_1^{(n_1)}, p_2^{(n_2)}, \ldots, p_k^{(n_k)}) \tag{1}$$

which means that p consists of p_1 (n_1 times), followed by p_2 (n_2 times) etc, where $p_1 < p_2 < \cdots < p_k$. In this notation, if $n_i = 1$, we may omit the superscript.

The possible arrival times during an hour is the set $\mathbf{Z}_{60} = \{0, 1, \ldots, 59\}$ which we view as the cyclic group equipped with addition modulo 60 (see e.g. [9], [16]). Let $0 \leq s < q$ be two integers and define

$$C_q^s = \{x \in \mathbf{Z}_{60} : x \equiv s \pmod{q}\}.$$

So $C_q^0, \ldots, C_q^{q-1}$ are the congruence classes modulo q in $\mathbf{Z}_{60}$. We call C_q^s a

q-*class* (or simply a *class*), and it corresponds to the arrival times of a line with period q where the "initial" train arrives at time s. For instance, $C_{10}^3 = \{3, 13, 23, 33, 43, 53\}$. A q-class will usually be denoted by C_q (so $C_q = C_q^s$ for some s).

We say that a period-vector $p \in \mathcal{P}^n$ is *admissible* if there are pairwise disjoint p_i-classes ($i \le n$), i.e., there are integers $0 \le s_i < p_i$ ($i \le n$) such that the classes $C_{p_1}^{s_1}, \ldots, C_{p_n}^{s_n}$ are pairwise disjoint. Such a family of p_i-classes will be called a *schedule*. The main problem we consider is:

(LS) *For given $p \in \mathcal{P}^n$ decide if p is admissible and, if so, find a corresponding schedule.*

We call this the *line scheduling problem* and denote it by *(LS)*. This is a combinatorial packing problem where the main difficulty is to construct *disjoint* p_i-classes, i.e., to avoid that more than one train arrives at any given time (minute). In particular, this corresponds to a minimum headway (distance) of precisely one minute. In the sequel we relax this assumption and consider the more general situation where the headway may be larger than one.

The following set is also of interest

$$\begin{aligned} \mathcal{N}(p_1, p_2, \ldots, p_k) = {} & \text{the set of maximal vectors } (n_1, n_2, \ldots, n_k) \text{ such} \\ & \text{that } p = (p_1^{(n_1)}, p_2^{(n_2)}, \ldots, p_k^{(n_k)}) \text{ is admissible.} \end{aligned} \tag{2}$$

Here "maximal" refers to the componentwise ordering. Thus, a vector p of the form (1) is admissible if and only if $\mathcal{N}(p_1, p_2, \ldots, p_k)$ contains a vector $(m_1, m_2, \ldots, m_k)$ with $n_i \le m_i$ ($i \le k$).

Note: each line i with $p_i = 1$ or $p_i = 60$ is easy to handle in LS, so we assume hereafter that p contains no such components.

Literature. There exists an extensive amount of research on different kinds of timetabling and scheduling problems. These are important problems in the transportation field where trains or metro lines are to be scheduled in networks under various constraints and goals concerning the timetables. As mentioned realistic railway timetabling models and methods are discussed in [6]. In particular, in that volume, the paper [14] considers the integration of line planning, timetabling and vehicle scheduling into a periodic scheduling problem, and proposes promising solution methods based on mixed integer programming. In the proceedings [10] from ATMOS 2006, the 6th Workshop on Algorithmic Methods and Models for Optimization of Railways, there are several papers on planning and scheduling problems for railways. In particular, the paper [1] introduces the following problem, called the *periodic metro scheduling* problem (PMS): given a rail network, a set of routes (paths in this network) and a time period, one seeks departure times of the routes so that one maximizes the minimum time between any two trains passing from the same point of the network. This goal is chosen to obtain a delay-tolerant transportation system. In PMS the travel times along edges in the network are taken into account. The LS problem of this paper may be seen as a special case of PMS where the network and the routes are very special (the routes only share a single edge). PMS is an optimization problem while we, in LS, only consider the *feasibility* problem: can a set of lines be scheduled or not. In [1] it is shown that PMS is *NP*-hard, even for ring networks, and different approximation algorithms, with given performance ratio, are presented. A purpose of this paper is to use the specific structure of the LS problem to give characterizations of when feasible schedules exist, and even to find explicit all feasible schedules in some cases.

Another interesting, and relevant, paper is [2] where one considers the problem of finding schedules for periodically recurring events, in particular for two events. This problem is elegantly presented in a geometrical manner by representing each regular recurring event by a regular polygon with vertices on a circle. The circle correspond to a unit time (e.g. 60 minutes), and the vertices correspond to the events. For instance, a polygon with 6 vertices may correspond to a (subway) line with period 10. Given two positive numbers m_1 and m_2 the goal is to find two regular polygons with m_1 and m_2 vertices in order to optimize some function of the distances $x_1, x_2, \ldots, x_{m_1+m_2}$ between consecutive vertices. A main theorem in [2] says that there is a pair of such polygons which, simultaneously, is optimal with respect to a whole family of interesting objective functions; including minimizing the maximal waiting time ($\min \max_k x_k$), and maximizing the safety interval ($\max \min_k x_k$). A few results for scheduling of three or more events were also given in [2]. We may view the scheduling problems in [2] as *continuous* optimization versions of the LS problem. The polygons can be placed arbitrarily on the circle so feasibility questions are of no interest for that problem. This is in contrast to LS where we have the integrality requirement, the arrival times (vertices of polygons) must be in the integers 0, 1, . . . , 59. We return to a connection between [2] and the LS problem in Section 3. The scheduling problem in [2] was generalized in [3] to consider irregular polygons (so the waiting times between consecutive trains on the same line may differ), again with a similar class of objective functions. This leads to a nonlinear nonconvex optimization problem which was shown in [3] to be *NP*-hard (for each of the objective functions) by a reduction from the 3-partitioning problem. Moreover, the authors developed a computational method by decomposing the problem into smaller subproblems, each being a convex optimization problem (with linear constraints).

We also mention that related problems to LS arise in other areas (class-teacher timetabling [5], exam schedules, work shift plans etc.). We refer to the survey paper [4] and the book [17] or [12], [13] for a discussion of such problems, models and algorithms, and also for further references in this area. Recent work with true implementations of optimization based solutions in railway systems was done by the team Kroon, Huisman, Abbink, Fioole, Fischetti, Maroti, Schrijver, Steenbeek, Ybema, and for this they received the Franz Edelman Award for Management Science Achievement ("The New Dutch Timetable: The OR Revolution"), see a forthcoming paper [11] in the INFORMS journal *Interfaces*. Also, related work on real-world railway timetabling has recently been done by C. Liebchen and is reported in [15].

The LS problem is of a number-theoretic nature and we have found some previous work in this direction. In [7] it was proved that pairwise disjoint congruence classes exist under a certain assumption, namely that the greatest common divisors of pairs among the numbers p_i ($i \leq n$) are different, and also different from 1. (They uses the term *harmonic n-tuple* corresponding to our notion of admissible vector.) The main result of [7] was later generalized in [19] where it was shown that if the number of pairs with the same greatest common divisor d is "sufficiently small" (for each d), then the given n-tuple is harmonic. In this paper the author also remarked that the problem of characterizing the harmonic n-tuples seems hard.

Organization. The remaining part of this paper is organized as follows. In Section 2 we give some basic, and useful, results while our main results are stated in Section 3. A general computational method for LS is found in Section 4. We use the standard notation $p|q$ to indicate that p is a divisor of q (so $q = kp$ for some integer k). The greatest common divisor of two integers a and b is denoted by $\gcd(a, b)$. For vectors $u, v \in \mathbb{R}^n$ we write $u \leq v$ for the componentwise ordering where $u_i \leq v_i$ ($i \leq n$).

2 Some basic results

The *frequency* of a period-vector $p \in \mathcal{P}^n$ is defined as the sum of its frequencies $\text{freq}(p) = \sum_{i=1}^{n} f_i$ (where $f_i = 60/p_i$). A simple necessary condition for a vector p to be admissible is stated next.

Lemma 1 *If $p \in \mathcal{P}^n$ is admissible, then* $\text{freq}(p) \leq 60$.

Proof. If p is admissible it has a schedule and $\text{freq}(p)$ is the total number of arrivals during an hour. □

We say that p is *complete* if $\text{freq}(p) = 60$. Later we give examples showing that the condition in Lemma 1 is far from being sufficient for p to be admissible.

The next lemma will be used repeatedly and it characterizes when two classes intersect.

Lemma 2 *Consider integers $0 \leq s < p$ and $0 \leq t < q$. Then the classes C_p^s and C_q^t intersect if and only if $s \equiv t \pmod{d}$ where $d = \gcd(p, q)$.*

Proof. Since $d = \gcd(p, q)$ there are integers a and b with $ad = p$ and $bd = q$. Assume first that $C_p^s \cap C_q^t$ is nonempty, and let $x \in C_p^s \cap C_q^t$. So $x = s + kp = t + lq$ for suitable integers k and l. Therefore $s + kad = t + lbd$ and $s \equiv t \pmod{d}$.

Conversely, assume that $s \equiv t \pmod{d}$, so $s - t = kd$ for some integer k. It follows from the Euclidean algorithm (for finding $d = \gcd(p, q)$) that there are integers a and b such that $d = ap + bq$ (see e.g. [16], [18]). Therefore, $s - t = kd = k(ap + bq)$ and $s - (ka)p = t + (kb)q$. This shows that $C_p^s \cap C_q^t$ is nonempty, and the proof is complete. $\square$

Concerning Lemma 2 we are mainly interested in the situation where p and q are divisors of 60 (periodic schedules). In that situation two congruence classes intersect if and only if they intersect in $\mathbf{Z}_{60}$. Thus, the lemma shows that we can test if two classes C_p^s and C_q^t intersect simply by performing one division, namely $(s - t)/d$ where $d = \gcd(p, q)$. For instance, C_{10}^2 and C_{15}^7 intersect as $\gcd(10, 15) = 5$ and $2 \equiv 7 \pmod{5}$. Indeed, we have $C_{10}^2 = \{2, 12, 22, 32, 42, 52\}$ and $C_{15}^7 = \{7, 22, 37, 52\}$ so they intersect in 22 and 52.

Based on Lemma 2 we can reformulate the LS problem as a problem of finding integers $0 \leq s_i < p_i$ $(i \leq n)$ such that

$$\begin{aligned} 0 \leq s_i < p_i & \quad (i \leq n), \\ s_i \not\equiv s_j \pmod{d_{ij}} & \quad (i, j \leq n,\ i \neq j) \end{aligned} \tag{3}$$

where $d_{ij} = \gcd(p_i, p_j)$. We use a closely related model in Section 5 to solve a generalized version of problem LS.

The following corollary contains a criterion for a period-vector to be inadmissible.

Corollary 3 *If $p \in \mathcal{P}^n$ is admissible, then $\gcd(p_i, p_j) > 1$ for all $i, j \leq n$ $(i \neq j)$.*

Proof. Assume that $d = \gcd(p_i, p_j) = 1$ for some pair i, j. Then, by Lemma 2, the classes $C_{p_i}^{s_i}$ and $C_{p_j}^{s_j}$ intersect for *all* s_i and s_j as $s_i \equiv s_j \pmod{1}$ holds. But this shows that p is inadmissible. $\square$

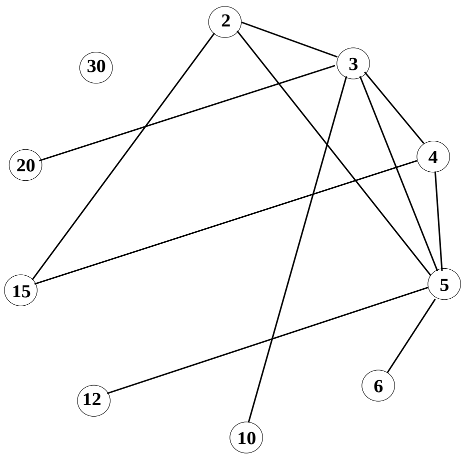

Fig. 1. Relative primes in $\mathcal{P}$.

Remark. Corollary 3 may also be deduced from a result in [7]: a necessary condition for $p \in \mathcal{P}^n$ to be admissible is that $\sum_{k \in R} 1/\tilde{p}_k(R) \leq 1$ for each $R \subseteq \{1, 2, \ldots, n\}$. Here $\tilde{p}_k(R) = \gcd(p_k, q_k)$ where q_k is the least common multiple of p_j, $j \in R \setminus \{k\}$. Thus, to prove Corollary 3, assume that $\gcd(p_i, p_j) = 1$ and let $R = \{i, j\}$. Then $\tilde{p}_i(R) = \tilde{p}_j(R) = 1$ so $\sum_{k \in R} 1/\tilde{p}_k(R) = 2 > 1$ and p must be inadmissible.

Corollary 3 says that any admissible period-vector p does not contain two components that are relatively prime. Figure 2 shows a graph where vertices correspond to $\mathcal{P}$ (except 1 and 60; see remark above) and edges indicate relatively prime pairs. Thus the set $\{p_i : i \leq n\}$ must be a stable set (independent set) in this graph when p is admissible. For instance, this shows that all the following period-vectors are inadmissible: (i) $(3, 4)$, (ii) $(5, 12)$, (iii) $(2, 15)$, and (iii) $(3, 20)$.

The next corollary generalizes the idea in Corollary 3.

Corollary 4 *Let $p \in \mathcal{P}^n$ be admissible. Then, for each positive integer d, the vector p contains at most d components such that $\gcd(p_i, p_j) = d$ for each pair i, j among these components.*

Proof. Let $I \subseteq \{1, 2, \ldots, n\}$ be such that $\gcd(p_i, p_j) = d$ for each pair $i, j \in I$. Consider a schedule for p (which exists as p is admissible), and let line i be assigned to the class $C_{p_i}^{s_i}$, $i \in I$. Now, for each $i \in I$, s_i is congruent modulo d to some number in $\{0, 1, \ldots, d-1\}$. Since $\gcd(p_i, p_j) = d$ for each pair $i, j \in I$, it follows from Lemma 2 that $s_i \not\equiv s_j \pmod d$ for each pair $i, j \in I$, and therefore $|I| \leq d$. □

Example 1 *By Corollary 4 all of the following vectors are inadmissible: (i)* $p = (2, 10, 12)$ *(using* $d = 2$*), (ii)* $p = (5^{(4)}, 10^{(1)}, 15^{(1)})$ *(using* $d = 5$*) and (iii)* $p = (q^{(q+1)})$ *for some* q *(using* $d = q$*). In the last case* $\mathrm{freq}(p) = 60(q+1)/q > 60$ *so it also follows from Lemma 1 that* p *is inadmissible.*

This example also shows that p may be inadmissible even if $\mathrm{freq}(p)$ is much smaller that 60. For instance, $p = (2, 10, 12)$ is inadmissible, but $\mathrm{freq}(p) = 41$. So it is (somehow) the internal structure of the greatest common divisors of the periods p_i that matter.

We shall also need the following simple result.

Lemma 5 *Let* $d|p$ *and consider a* p*-class* C_p *and a* d*-class* C_d *that intersect. Then* $C_p \subseteq C_d$*.*

Proof. Let $y \in C_p \cap C_d$. Then C_p consists of all integers x (in $\mathbf{Z}_{60}$) that are congruent to y modulo p. But each such number x is also congruent to y modulo d, as $d|p$. Thus, $C_p \subseteq C_d$. □

3 A characterization and an extension

It would be nice to have a characterization of admissible period-vectors in LS in its complete generality. We have not been able to establish such a result, but the following theorem contains a characterization which covers a large class of instances of LS.

Theorem 6 *Let* $p \in \mathcal{P}^n$ *be a period-vector of the form*

$$p = (p_1^{(n_1)}, p_2^{(n_2)}, \ldots, p_k^{(n_k)}). \tag{4}$$

Assume that $p_i = dq_i$ $(i \leq k)$ *for some integer* d *and that each pair of numbers among* $q_1, q_2, \ldots, q_k$ *are relative prime. Then* p *is admissible if and only if*

$$\sum_{i=1}^{k} \lceil n_i/q_i \rceil \leq d. \tag{5}$$

The set $\mathcal{N}(p_1, p_2, \ldots, p_k)$ *consists of all vectors* $(m_1q_1, m_2q_2, \ldots, m_kq_k)$ *where* $m_1, m_2, \ldots, m_k$ *are nonnegative integers with* $\sum_{i=1}^{k} m_i = d$*. Moreover, if (5) holds, a schedule for* p *is obtained by assigning lines with period* p_i *to* $\lceil n_i/q_i \rceil$ *congruence classes modulo* d *(each can take* q_i *lines) for* $i = 1, 2, \ldots, k$*.*

Proof. Let p be of the form in (4). Assume first that p is admissible. We may clearly assume that each $n_i \geq 1$. Consider a schedule with classes $C^{(1)}, C^{(2)}, \ldots, C^{(n)}$ associated with p.

Claim: $C^{(i)}$ is contained in some d-class ($i \leq n$). Moreover, the lines assigned to the same d-class all have the same period.

Proof of Claim: A line with period p_i is assigned to a class $C^s_{p_i}$ for some s, and since $d|p_i$ we conclude by Lemma 5 that $C^s_{p_i}$ is contained in some d-class C_d. Note that there are d distinct such d-classes. Consider now two lines with different periods p_i and p_j, so $i \neq j$; say that these lines are assigned to the classes $C^s_{p_i}$ and $C^t_{p_j}$, respectively. Since $\gcd(p_i, p_j) = d$ and $C^s_{p_i} \cap C^t_{p_j} = \emptyset$, we have $s \not\equiv t \pmod{d}$, by Lemma 2. So s and t are congruent to *different* numbers in $\{0, 1, \ldots, d-1\}$ (modulo d). The Claim now follows.

Consider a d-class C^s_d which contains a p_i-class for some $i \leq k$. This class C^s_d will only contain p_i-classes (by the Claim), and the maximum number of these p_i-classes contained in C^s_d is $p_i/d = q_i$. The number of d-classes required to assign all the n_i lines with period p_i is equal to $\lceil n_i/q_i \rceil$. But since the total number of d-classes is d the following inequality must hold

$$\sum_{i=1}^{k} \lceil n_i/q_i \rceil \leq d.$$

This proves (5). It is also clear from these arguments that if (5) holds then p is admissible and that an associated schedule exists with the form described in the theorem. Finally, the structure of $\mathcal{N}(p_1, p_2, \ldots, p_k)$ also follows from the first part of the theorem (m_i is the number of d-classes containing p_i-classes), and the proof is complete. $\square$

Some comments to this theorem are given next:

(1) The theorem does not use the assumption that p_i is a divisor of 60, so it holds for arbitrary integers p_i (of the mentioned form).
(2) The number d occurring in this theorem is the greatest common divisor of the components of p. The assumption on p in Theorem 6 means that $\gcd(p_i, p_j) = d$ for all $i \neq j$.
(3) An attractive feature of period-vectors of the form stated in Theorem 6 is that the corresponding time schedules have a simple structure.
(4) Consider a period-vector p corresponding to $\mathcal{N}(p_1, p_2, \ldots, p_k)$, where p_1, $p_2, \ldots, p_k$ is as described in the theorem. Then $\mathrm{freq}(p) = 60$, so p is complete.

Example 2 *Let $p = (2^{(n_1)}, 4^{(n_2)}, 6^{(n_3)}, 10^{(n_4)})$, so p has the form (4) with $d = 2$, $q_1 = 1$, $q_2 = 2$, $q_3 = 3$ and $q_4 = 5$. Here the q_i's are pairwise relatively* *prime. Thus, by Theorem 6, p is admissible if and only if*

$$n_1 + \lceil n_2/2 \rceil + \lceil n_3/3 \rceil + \lceil n_4/5 \rceil \leq 2.$$

In particular, at most two of the n_i's are positive when p is admissible. For instance, both $p = (2^{(0)}, 4^{(2)}, 6^{(0)}, 10^{(5)})$ and $p = (2^{(0)}, 4^{(0)}, 6^{(3)}, 10^{(5)})$ are admissible, while $p = (2^{(0)}, 4^{(1)}, 6^{(1)}, 10^{(1)})$ is inadmissible (and here $\mathrm{freq}(p) = 15 + 10 + 6 = 31$).

Example 3 *Let $p = (5^{(n_1)}, 10^{(n_2)}, 15^{(n_3)})$. This vector has the form (4) with $d = 5$, $q_1 = 1$, $q_2 = 2$ and $q_3 = 3$, and these q_i's are pairwise relatively prime. By Theorem 6, p is admissible if and only if*

$$n_1 + \lceil n_2/2 \rceil + \lceil n_3/3 \rceil \leq 5.$$

From this we see, for instance, that $p = (5^{(1)}, 10^{(4)}, 15^{(6)})$ is admissible while $p = (5^{(1)}, 10^{(3)}, 15^{(7)})$ is inadmissible.

Example 4 *Let $p = (4^{(n_1)}, 12^{(n_2)}, 20^{(n_3)})$, so again p has the desired form with $d = 4$, $q_1 = 1$, $q_2 = 3$ and $q_3 = 5$. These q_i's are pairwise relatively prime so, by Theorem 6, p is admissible if and only if*

$$n_1 + \lceil n_2/3 \rceil + \lceil n_3/5 \rceil \leq 4.$$

So, for instance, $p = (4^{(2)}, 12^{(3)}, 20^{(5)})$ is admissible, and $\mathrm{freq}(p) = 60$.

We now consider the special case of Theorem 6 where $k \leq 2$, i.e., there are at most two different periods. The simplest case is $k = 1$, say $p = (p_1^{(n_1)})$ for some $p_1 \in \mathcal{P}$. Then p is admissible if and only if $n_1 \leq p_1$ (confer Example 1). Moreover, when $n_1 \leq p_1$ a schedule is obtained by allocating line i to the class $C_{p_1}^i$ $(0 \leq i < n_1)$.

Example 5 *By January 2007 the subway network in Oslo consists of 6 lines, each with period 15, and the time schedule used is shown in the following table for the first quarter (the schedule repeats in each of the next quarters)*

minute	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
$h(\cdot)$	*0*	*0*	*1*	*0*	*2*	*0*	*0*	*3*	*0*	*0*	*4*	*0*	*5*	*0*	*6*

(We have reordered the line numbers, for simplicity). Here, as in the general situation, there may be several different schedules for an admissible period-vector p. The schedule above has the desired property that the subsequences of zeros have as even length as possible. This reduces the consequences of

possible delays. Actually, one wants a distance of at least two minutes between consecutive arrivals; we consider this extended problem in detail below.

Next, we consider the case $k = 2$, so there are two different periods.

Corollary 7 *Let $p = (p_1^{(n_1)}, p_2^{(n_2)})$. Define $d = \gcd(p_1, p_2)$ and $q_1 = p_1/d$, $q_2 = p_2/d$. Then p is admissible if and only if $\lceil n_1/q_1 \rceil + \lceil n_2/q_2 \rceil \leq d$. Moreover, the set $\mathcal{N}(p_1, p_2)$ is given by*

$$\mathcal{N}(p_1, p_2) = \{(rq_1, (d-r)q_2) : 0 \leq r \leq d,\ r \in \mathbf{Z}\}.$$

A schedule for an admissible period-vector p is obtained by assigning lines with period p_1 to r congruence classes C_d (each can take q_1 lines) and the remaining lines to the remaining $d - r$ congruence classes C_d (each can take q_2 lines).

Proof. Let $k = 2$ in Theorem 6. □

Example 6 *Let $p_1 = 10$, $p_2 = 15$. Then $d = \gcd(10, 15) = 5$ and $q_1 = 10/5 = 2$, $q_2 = 15/5 = 3$. We then find that*

$$\mathcal{N}(10, 15) = \{(0, 15), (2, 12), (4, 9), (6, 6), (8, 3), (10, 0)\}.$$

Thus, we have completely solved the case when there are at most two periods. Moreover, the corresponding schedules are very simple as discussed in Corollary 7.

In the previously mentioned paper [2] (see Section 1) there are some results that are related to Corollary 7. Proposition 3 in [2] says that n_1 (regular) polygons with m_1 vertices and n_2 (regular) polygons with m_2 vertices can be arranged on a circle (of unit length) so that the minimum length between adjacent vertices is $1/M$. Here $M = 2\,\mathrm{lcm}(m_1, m_2)$ (where lcm denotes least common multiple), $n_1 = m_2/d_0$, $n_2 = m_1/d_0$ and $d_0 = \gcd(m_1, m_2)$. The construction here is to place M equidistant points on the circle, and use the even numbered vertices for the polygons with m_1 vertices, and the odd numbered vertices for all the other polygons. To relate to LS we assume the circle corresponds to 60 minutes, and that $m_1|60$ and $m_2|60$. Let $p_i = 60/m_i$ $(i = 1, 2)$ be the two periods. Then all the equidistant vertices correspond to integers among $0, 1, \ldots, 59$ (assuming, as we may, that one vertex corresponds to 0) if and only if $60/M$ is an integer. This, again, is equivalent to the condition that both p_1 and p_2 are even numbers. The construction of Proposition 3 then gives a feasible schedule in LS and $(n_1, n_2) \in \mathcal{N}(p_1, p_2)$. Note, however, that there are other feasible schedules than this one, and the complete set of feasible schedules is described in Corollary 7.

In the remaining part of this section we consider a generalization of Corollary 7 which reflects stronger safety distance than 1 minute between consecutive trains. First, we discuss a suitable notion of distance in this setting.

Given a positive integer d, we define a certain distance $\delta_d(x,y)$ between two integers x, y by $\delta_d(x,y) := \min\{|\tilde{x}-\tilde{y}|, d-|\tilde{x}-\tilde{y}|\}$ where $\tilde{x}$ (resp. $\tilde{y}$) denotes the remainder of x (resp. y) divided by d. This is the distance between $\tilde{x}$ and $\tilde{y}$ on a cycle of length d. For instance, $\delta_{60}(3,58) = 5$ and $\delta_5(9,16) = \delta_5(4,1) = 2$. Moreover, we define the distance between two classes C_p^s and C_q^t by $\delta(C_p^s, C_q^t) = \min\{\delta_{60}(x,y) : x \in C_p^s,\ y \in C_q^t\}$. So, if two lines use classes C_p^s and C_q^t respectively, then $\delta(C_p^s, C_q^t)$ is the shortest distance between two consecutive trains from different lines.

The following lemma generalizes Lemma 2 and shows how to calculate the distance between two classes in a simple way.

Lemma 8 *Consider integers $0 \le s < p$ and $0 \le t < q$, and let $d = \gcd(p,q)$. Then $\delta(C_p^s, C_q^t) = \delta_d(s,t)$.*

Proof. Since p and q are multiples of $d = \gcd(p,q)$, it follows that if $x \in C_p^s$ and $y \in C_q^t$ then $x - y = s - t - md$ for some integer m. Thus

$$\begin{aligned}\delta(C_p^s, C_q^t) &= \min\{\delta_{60}(x,y) : x \in C_p^s,\ y \in C_q^t\} \\ &\ge \min\{\delta_{60}(x,y) : x - y = s - t - md,\ m \in \mathbf{Z}\} \qquad (6) \\ &= \delta_d(s,t).\end{aligned}$$

We verify the last equality. The minimizing m is either $\lfloor s/d \rfloor - \lfloor t/d \rfloor$ or $\lfloor s/d \rfloor - \lfloor t/d \rfloor + 1$; and the desired equality holds. It remains to prove that the inequality in (6) can be attained. As mentioned in the proof of Lemma 2 there are integers a and b such that $d = ap + bq$. Then it is clear that equality is attained in (6) by letting $x = s - map$ and $y = t + mbq$ where m is the minimizer in (6); for then $x - y = s - t - m(ap + bq) = s - t - md$ as desired. □

We define a period-vector $p \in \mathcal{P}^n$ to be *ν-admissible* if it has a schedule with classes $C^{(1)}, C^{(2)}, \ldots, C^{(n)}$ such that $\delta(C^{(i)}, C^{(j)}) \ge \nu$ for all $i, j \le n$, $i \ne j$. We also define $\mathcal{N}_\nu(p_1, p_2, \ldots, p_k)$ as the set of maximal vectors $(n_1, n_2, \ldots, n_k)$ such that $p = (p_1^{(n_1)}, p_2^{(n_2)}, \ldots, p_k^{(n_k)})$ is ν-admissible.

Theorem 9 *Let $\nu \ge 1$ be an integer.*

(i) Let $p = (p_1^{(n_1)})$. Then p is ν-admissible if and only if $n_1 \le \lfloor p_1/\nu \rfloor$.

(ii) Let $p = (p_1^{(n_1)}, p_2^{(n_2)})$ where $n_1, n_2 \ge 1$. Define $d = \gcd(p_1, p_2)$, $q_1 = p_1/d$,

and $q_2 = p_2/d$. *Then* p *is* ν*-admissible if and only if* $\lceil n_1/q_1 \rceil + \lceil n_2/q_2 \rceil \leq \lfloor d/\nu \rfloor$. *Moreover, the set* $\mathcal{N}_\nu(p_1, p_2)$ *is equal to*

$$\{(\lfloor p_1/\nu \rfloor, 0), (0, \lfloor p_2/\nu \rfloor)\} \cup \{(rq_1, (\lfloor d/\nu \rfloor - r)q_2) : 0 < r < \lfloor d/\nu \rfloor,\ r \in \mathbf{Z}\}.$$

Proof. By Lemma 8 we cannot allocate a p_1-line and a p_2-line to two classes modulo d with δ_d-distance less than ν. Moreover, ν consecutive classes modulo d cannot contain more than q_1 lines with period p_1; for then the δ_d-distance would be less than ν. Thus a maximal vector is obtained by using classes C_d^i for $i = 0, \nu, 2\nu, \ldots, \lfloor d/\nu \rfloor$ and fill each of these classes with p_1-classes only, or with p_2-classes only. This proves the result. □

Note that in this theorem case (ii) only applies when both n_1 and n_2 are positive. We also note that the proof above describes a feasible schedule for a ν-admissible period-vector p (where at most two different periods are considered).

Example 7 *The following example is of interest in the study of the subway network in Oslo:* $p_1 = 10$, $p_2 = 15$ *and* $\nu = 2$. *So the minimum distance between consecutive trains is 2 minutes. We then obtain* $d = 5$, *and* $\lfloor d/\nu \rfloor = 2$. *So, using Theorem 9, we calculate the following maximal* 2*-admissible period-vectors:* $(5, 0)$, $(2, 3)$, *and* $(0, 7)$.

In practice it is of interest to compare different candidate period-vectors in terms of their "efficiency". We here consider a natural efficiency measure which corresponds to the number of trains arriving at the Central Station. For a ν-admissible period-vector $p \in \mathcal{P}^n$ we define its *density* (or rather, ν*-density*) by

$$\mathrm{dens}_\nu(p) = \frac{\mathrm{freq}(p)}{\frac{60}{\nu}} = \frac{\nu \cdot \mathrm{freq}(p)}{60}.$$

This is the proportion of the possible time slots (during an hour) that are used by p. Note that $0 \leq \mathrm{dens}_\nu(p) \leq 1$ as a ν-admissible period-vector cannot give more than $60/\nu$ arrivals during an hour. It is of interest to compare the density of different ν-admissible period-vectors. The following result shows the densities of all maximal ν-admissible period-vectors in the situation discussed in Theorem 9. (Clearly, non-maximal such period-vectors will have smaller density).

Theorem 10 *Consider the situation given in Theorem 9. For* $i = 1, 2$, *if* $p = (p_i^{(n_i)})$ *where* $n_i = \lfloor p_i/\nu \rfloor$, *then* $\kappa_i := \mathrm{dens}_\nu(p) = \lfloor p_i/\nu \rfloor \nu / p_i$. *If* $p = (p_1^{(n_1)}, p_2^{(n_2)})$ *where* $n_1 = rq_1$, $n_2 = (\lfloor d/\nu \rfloor - r)q_2$ *and* $0 < r < \lfloor d/\nu \rfloor$, *then* $\kappa_3 := \mathrm{dens}_\nu(p) = \lfloor d/\nu \rfloor \nu / d$ *(independent of* r*). Moreover, these densities* *satisfy* $\kappa_3 \leq \min\{\kappa_1, \kappa_2\}$.

Proof. Concerning $p = (p_1^{(n_1)}, p_2^{(n_2)})$ we use that $q_i/p_i = 1/d$ and obtain $\text{freq}(p) = rq_1 \cdot (60/p_1) + (\lfloor d/\nu \rfloor - r)q_2 \cdot (60/p_2) = 60r(1/d) + (\lfloor d/\nu \rfloor - r)60/d = (60/d)\lfloor d/\nu \rfloor$. This gives the desired density, and the other two cases are even easier to establish. To prove the last inequality, let $d = a\nu + r$ for integers a and r with $0 \leq r < \nu$. Then $\lfloor d/\nu \rfloor = a$. Also, $p_1 = q_1 d = aq_1\nu + rq_1$, so $\lfloor p_1/\nu \rfloor = \lfloor aq_1 + (rq_1/\nu) \rfloor \geq aq_1$. This gives

$$\kappa_1 = \lfloor p_1/\nu \rfloor \nu/p_1 \geq aq_1 \cdot \nu/p_1 = a\nu/d = \lfloor d/\nu \rfloor \nu/d = \kappa_3$$

Similarly we obtain $\kappa_2 \geq \kappa_3$ and the proof is complete. □

We illustrate the contents of Theorem 10 by the following example.

Example 8 *Consider the periods* $5, 6, 10$ *and* 15. *The following table shows the densities for some corresponding period-vectors. Each row corresponds to a fixed value of* ν. *The column denoted by* 5 *shows* $\text{dens}_\nu((5^{(n_1)}))$ *where* $n_1 = \lfloor 5/\nu \rfloor$. *The next columns, denoted by* $6, 10$ *and* 15, *show the corresponding densities for each of these maximal period-vectors. The last six columns show the densities for (positive) maximal mixed period-vectors of the form* $p = (p_1^{(n_1)}, p_2^{(n_2)})$. *Note that there are only such vectors if* $d = \gcd(p_1, p_2) \geq 2\nu$, *confer Theorem 10.*

ν	5	6	10	15	(5,6)	(5,10)	(6,10)	(5,15)	(6,15)	(10,15)
1	1.00	1.00	1.00	1.00	0.00	1.00	1.00	1.00	1.00	1.00
2	0.80	1.00	1.00	0.93	0.00	0.80	0.00	0.80	0.00	0.80
3	0.60	1.00	0.90	1.00	0.00	0.00	0.00	0.00	0.00	0.00

Fig. 2. Comparison of densities

As this example shows, the density typically decreases very fast when ν (headway) is increased, in particular for mixed period-vectors.

Finally, in practice, it may be required to construct schedules with shorter headway with, say, possible arrivals every 30 seconds. All our results may be adopted to this situation by considering a time span of 120 time units.

4 A computational approach for LS

In this last section we present a computational method for solving LS in the general case. This method is based on integer linear programming.

The idea is to consider an optimization version of problem LS where the goal is to decide for each line if it should be used or not. The objective function is to maximize the total number of arrivals (at the Central Station), i.e. the sum of the frequencies of the selected lines. Consider the following zero-one ILP (integer linear programming) problem:

$$
\begin{array}{lll}
\max & \sum_{i,k} x_{ik} & \\
\text{subject to} & & \\
\text{(i)} & \sum_i x_{ik} \leq 1 & \text{for each } k \\
\text{(ii)} & \sum_{j=k}^{k+p_i-1} x_{ik} \leq 1 & \text{for each } i,\ k \\
\text{(iii)} & x_{ik} = x_{ij} & \text{when } k \equiv j \pmod{p_i} \\
\text{(iv)} & x_{ik} \in \{0,1\} & \text{for all } i,\ k.
\end{array}
\tag{7}
$$

In this model the binary variable x_{ik} is set to 1 if line i is arriving at time k ($i \leq n$, $k \leq 60$). Constraints (i) say that at most one line can arrive at time k. Constraints (ii) reflect that during a time period of p_i minutes line i arrives at most once, and (iii) are the periodicity constraints: if line i is used, it arrives every p_i minutes, starting from some time k with $0 \leq k < p_i$. Note that equality holds in constraint (ii) whenever line i is used.

The ILP problem (7) may be extended to find a ν-admissible schedule (using the same objective function). This is done by replacing constraints (i) by

$$
\text{(i')} \qquad \sum_{i \leq n,\, p \in I(k;\nu)} x_{ip} \leq 1 \quad \text{for each } k
$$

where $I(k;\nu) = \{k, k+1, \ldots, k+\nu-1\}$ where these indices are calculated modulo 60.

It turns out that problem (7) can be solved very fast (actually, in less than a second) by efficient codes for integer linear programming. We used ILOG CPLEX [8] for this; it is an extremely fast linear programming solver with a branch-and-bound algorithm for solving ILP's. All the examples reported below were solved in about 0.3 seconds on a standard laptop.

As a first example consider $n = 9$, $\nu = 1$ and

$$
p = (5, 5, 5, 10, 10, 15, 15, 30, 30).
$$

This problem was solved in the root node, i.e., no branching was required. The optimal solution x that was found is a $(0,1)$-matrix of size 9×60 and its first 30 columns is shown next (the schedule repeats for the next 30 minutes as each period is a divisor of 30):

$$\begin{bmatrix}
100001000010000100001000010000 \\
000100001000010000100001000010 \\
000010000100001000010000100001 \\
000000010000000001000000000100 \\
001000000000100000000010000000 \\
000000000001000000000000001000 \\
000000100000000000000100000000 \\
000000000000000010000000000000 \\
010000000000000000000000000000
\end{bmatrix}$$

There are no zero rows so all lines are in use, and the optimal value is therefore the sum of all frequencies which is 60, so the corresponding period-vector is complete. We also solved the same instance (i.e., the same p), but with $\nu = 2$. This took 80 branch-and-bound nodes, and the optimal value (the sum of all frequencies) was 24. The optimal solution is here to use two of the lines with $p_i = 5$. So, for $\nu = 2$, p is not complete as the sum of frequencies is less than 30.

Consider another example where

$$p = (5, 6, 6, 6, 10, 10, 12, 15, 15, 15, 30, 30).$$

First, let $\nu = 1$. Here $n = 12$ and $\text{freq}(p) = 75$ so p is inadmissible. However, the optimization problem still makes sense. In this case CPLEX found an optimal solution where lines with periods 6, 6, 6, 12, 15, 15, 15, 30, 30 were used, but the two lines with period 10 were not used. The optimal value is then 51. The problem required 63 branch-and-bound nodes. For the same period-vector p and with $\nu = 2$ we obtained (after 11 branch-and-bound nodes) an optimal solution where all lines with $p_i = 6$ were used, so the optimal value was 30. Thus, for $\nu = 2$, p is complete.

Based on extensive testing it is our experience that this rather straightforward approach to the optimization version of LS, based on integer linear programming, solves these problems extremely fast. Thus, from a practical point of view, it could be used in a planning process. Furthermore, as we discussed in Section 3, several instances of LS may actually be solved analytically using Theorem 6 directly or by a combinatorial discussion based on the theorem.

Acknowledgment. The author wishes to thank two referees for several very useful comments and suggestions, and for pointing out highly relevant literature. Moreover, they made useful suggestions in connection with Corollary 3 and Theorem 10. Also, many thanks to Harald Minken for interesting discussions about the topic.

References

[1] E. Bampas, G. Kaouri, M. Lampis, A. Pagourtzis, Periodic Metro Scheduling, In R. Jacob, M. Müller-Hannemann (Eds.) ATMOS 2006 - 6th Workshop on Algorithmic Methods and Models for Optimization of Railways, Dagstuhl Seminar Proceedings, Volume 06002, 2006.

[2] R.E. Burkard, Optimal schedules for periodically recurring events, Discrete Appl. Math. 15 (1986), no. 2-3, 167–180.

[3] P. Brucker, R.E. Burkard, J. Hurink, Cyclic schedules for r irregularly occurring events, J. Comput. Appl. Math. 30 (1990), no. 2, 173–189.

[4] D. de Werra, The combinatorics of timetabling, European Journal of Operational Research 96 (No. 3)(1997) 504–513.

[5] D. de Werra, Constraints of availability in timetabling and scheduling, in: E. Burke, P. De Causmaecker (Eds.), Practice and Theory of Automated Timetabling IV. Lecture Notes in Computer Science, Volume 2740/2003, Springer, 2003, pp. 3–23.

[6] F. Geraets, L.G. Kroon, A. Schöbel, D. Wagner and C.D. Zaroliagis (Eds.), Algorithmic Methods for Railway Optimization, Lecture Notes in Computer Science, Springer Berlin / Heidelberg, Volume 4359/2007.

[7] A.P. Huhn, L. Megyesi, On disjoint residue classes, Discrete Mathematics 41 (1982) 327–330.

[8] ILOG CPLEX (OPL Development Studio 5.5, April 2007), see http://www.ilog.com/products/cplex/

[9] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, Springer, New York, 1990.

[10] R. Jacob, M. Müller-Hannemann (Eds.) ATMOS 2006 - 6th Workshop on Algorithmic Methods and Models for Optimization of Railways, Dagstuhl Seminar Proceedings, Volume 06002, 2006. http://dblp.uni-trier.de

[11] L. Kroon, D. Huisman, E. Abbink, P. Fioole, M. Fischetti, G. Marti, A. Schrijver, A. Steenbeek, R. Ybema, Franz Edelman Award for Management Science Achievement (winner): The New Dutch Timetable: The OR Revolution, Forthcoming article in *Interfaces*.

[12] C. Liebchen, L. Peeters, On cyclic timetabling and cycles in graphs, Technical report, Technische Universität Berlin, No. 761/2002.

[13] C. Liebchen, L. Peeters, Some practical aspetc of periodic timetabling, In P. Chamoni et al: Operations Research 2001, Springer, 2002.

[14] C. Liebchen, R.H. Möhring, The Modeling Power of the Periodic Event Scheduling Problem: Railway Timetables - and Beyond, In F. Geraets et al. (Eds.), Algorithmic Methods for Railway Optimization, Lecture Notes in Computer Science, Springer Berlin / Heidelberg, Volume 4359/2007.

[15] C. Liebchen, The First Optimized Railway Timetable in Practice, *Transportation Science*, Vol. 42, Issue 4 (Nov. 2008).

[16] O. Ore, Number Theory and Its History, Dover Publ., New York, 1988. (First published in 1948 by McGraw-Hill).

[17] E. Burke, P. De Causmaecker (Eds.), Practice and Theory of Automated Timetabling IV. Lecture Notes in Computer Science, Volume 2740/2003. Springer, 2003.

[18] A. Schrijver, Theory of Linear and Integer Programming, Wiley, Chichester, 1986.

[19] Zhi-Wei Sun, On disjoint residue classes, Discrete Mathematics 104 (1992) 321–326.